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K_1-GROUPS, QUASIDIAGONALITY, AND INTERPOLATION BY MULTIPLIER PROJECTIONS

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ABSTRACT. We relate the following conditions on a σ-unital C^*-algebra A with the "FS property": (a) $K_1(A) = 0$; (b) every projection in $M(A)/A$ lifts; (c) the general Weyl-von Neumann theorem holds in $M(A)$: Any selfadjoint element h in $M(A)$ can be written as $h = \sum_{i=1}^{\infty} \lambda_i p_i + a$ for some selfadjoint element a in A, some bounded real sequence $\{\lambda_i\}$, and some mutually orthogonal projections $\{p_i\}$ in A with $\sum_{i=1}^{\infty} p_i = 1$; (d) $M(A)$ has FS; and (e) interpolation by multiplier projections holds: For any closed projections p and q in A^{**} with $pq = 0$, there is a projection r in $M(A)$ such that $p \leq r \leq 1 - q$.

We prove various equivalent versions of (a)–(e), and show that (e) $\Leftrightarrow$ (d) $\Leftrightarrow$ (c) $\Rightarrow$ (b) $\Leftarrow$ (a), and that (a) $\Leftrightarrow$ (b) if, in addition, A is stable. Combining the above results, we obtain counterexamples to the conjecture of G. K. Pedersen "A has $FS \Rightarrow M(A)$ has FS" (for example the stabilized Bunce-Deddens algebras). Hence the generalized Weyl-von Neumann theorem does not generally hold in $L(H_A)$ for σ-unital C^*-algebras with FS.

0. INTRODUCTION

Let K be the algebra of all compact operators on a separable Hilbert space H, and let $L(H)$ be the algebra of all bounded operators on H.

If A is a C^*-algebra, we denote the Banach space double dual of A by A^{**} and the multiplier algebra of A by $M(A)$. We have inclusions $A \subset M(A) \subset A^{**}$. For details of multiplier algebras the reader is referred to [3, 7, 11, 14, 24] among others.

Let $H_A = \{\{a_i\}: a_i \in A \text{ and } \sum_{i=1}^{\infty} a_i^* a_i \text{ converges in norm}\}$. Then H_A becomes a Hilbert (right) A-module with the A-valued inner product

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^{\infty} a_i^* b_i \quad \text{for all } \{a_i\}, \{b_i\} \in H_A.$$

We denote by $L(H_A)$ the set of all "bounded" module maps with an adjoint and by $K(H_A)$ the closed ideal of $L(H_A)$ called the "compact maps". More

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precisely, $K(H_A)$ is the norm closure of the set of all "finite rank" module maps with the form

$$\left\{\sum_{i=1}^{n} \theta_{x_i, y_i} : x_i, y_i \in H_A \text{ and } n \in N\right\}.$$

Here for any pair of elements x and y in H_A, $\theta_{x,y}$ is defined by $\theta_{x,y}(a) = x\langle y, a\rangle$ for $a \in H_A$. It is well known that

$$L(H_A) \cong M(A \otimes K) \quad \text{and} \quad K(H_A) \cong A \otimes K$$

as C^*-algebras (see [24] for details). The definitions of $L(H_A)$ and $K(H_A)$ are closely analogous to those of $\mathrm{L}(H)$ and K, but many aspects of this analogy are far from clear.

We call $L(H_A)/K(H_A)$ the generalized Calkin algebra, and more generally $M(A)/A$ the corona algebra of A. The canonical map from $M(A)$ to $M(A)/A$ is denoted by π.

The classical Kuiper's theorem has been generalized; that is, the unitary group of $L(H_A)$ is contractible in norm if A is σ-unital (see [17, 25]). One natural question is whether the analogue of the Weyl-von Neumann theorem is true in $L(H_A)$, or more generally in $M(A)$, for some C^*-algebras (for example AF algebras). We say that the generalized Weyl-von Neumann theorem holds in $M(A)$ if $M(A)$ has the following "Weyl-von Neumann property":

> For any selfadjoint element h of $M(A)$, there are a bounded real sequence $\{\lambda_i\}$ and mutually orthogonal projections $\{p_i\}$ in A such that $\sum_{i=1}^{\infty} p_i = 1$ and $h = \sum_{i=1}^{\infty} \lambda_i p_i + a$ for some selfadjoint element a in A. (We shall say that h is strongly quasidiagonal.)

We cannot expect a positive answer in a great generality due to the nature of the problem. We consider the Weyl-von Neumann property for the class of C^*-algebras with the "FS" property. A C^*-algebra A is said to have the FS property if the set of selfadjoint elements with finite spectrum is norm dense in the set of all selfadjoint elements of A [6, 26]. AF algebras, the Calkin algebra, von Neumann algebras, and Bunce-Deddens algebras [5; 8, §4] have the FS property. Various other examples of such C^*-algebras will be given in subsequent papers of the author [30]. It has recently been proved [13] that A has FS if and only if $A \otimes K$ has FS; again iff A has real rank zero. A C^*-algebra A is said to have the HP property if every hereditary C^*-subalgebra of A has an approximate identity consisting of projections. It is known that A has FS if and only if A has HP [26; 6, 2.7]. Recall that the K_1-groups of AF algebras and von Neumann algebras are trivial [7]. But the K_1-group is not necessarily trivial for a general C^*-algebra with FS. The Calkin algebra and Bunce-Deddens algebras provide well-known counterexamples [7, 9.3.1; 5; 8, §4; 7, 10.11.4].

This paper is arranged as follows. In §1 we give necessary preliminaries. In §2–§4, we assume that A is a C^*-algebra with FS (usually A is σ-unital).

In §2 we shall relate the lifting of projections from $M(A)/A$ to $M(A)$ to the triviality of K_1-groups and quasidiagonality of selfadjoint elements in $M(A)$ by proving various equivalent conditions. L. G. Brown [10] proved, as an application of the six-term exact sequence of K-theory, that every projection in $M(A)/A$ lifts if A is an app-w^*-algebra (this includes all AF algebras). G. A. Elliott gave a different proof of Brown's result for AF algebras [19]. Later M.-D. Choi generalized this result with an elementary proof [15]. We shall prove that if the generalized Weyl-von Neumann theorem holds in $M(A)$, then every projection in $M(A)/A$ lifts, and that every projection in $M(A)/A$ lifts if $K_1(A) = 0$. This generalizes L. G. Brown's result in [10] (also our proof is different). As a consequence, we obtain that if the Weyl-von Neumann theorem holds in $L(H_A)$, then $K_1(A) = 0$.

In §3 we prove the equivalence of five conditions to the strong quasidiagonality of a selfadjoint element in $M(A)$ if A is σ-unital. The equivalence of these conditions was partially established in [23]. Here we also give a different proof. Consequently, we obtain that the generalized Weyl-von Neumann theorem holds in $M(A)$ if and only if $M(A)$ has FS, and this implies that every projection in $M(A)/A$ lifts. Each of these conditions implies that $K_1(A) = 0$ if A is stable. Thus, the stabilized Bunce-Deddens algebras and the Calkin algebra provide basic counterexamples to the conjecture of G. K. Pedersen " A has $FS \Rightarrow M(A)$ has FS". Various nontrivial counterexamples for " $M(A)/A$ has FS but $M(A)$ does not" will be given in subsequent papers [30].

Any σ-unital C^*-algebra (not necessarily with FS) has "the interpolation by multipliers" property: If p and q are two mutually orthogonal closed projections in A^{**}, there is h in $M(A)$ such that $p \le h \le 1 - q$ (see [11]). If the above h can be replaced by a projection in $M(A)$, we say that "interpolation by multiplier projections" holds. L. G. Brown has recently proved that $M(A)$ has FS if and only if A has FS and interpolation by multiplier projections holds [12]. In §4 we give five conditions equivalent to the interpolation by multiplier projections in the hope that they will be of use in settling the problem whether $M(A)$ has FS if A is AF. Combining results of [12] and Theorems 3.1 and 4.2, we obtain eleven conditions equivalent to the Weyl-von Neumann property.

1. Preliminaries

First of all, we fix some notations. If A is a C^*-algebra, we denote the set of all selfadjoint elements of A by $A_{\text{s.a.}}$ and the set of all positive elements of A by A_+. If p and q are two projections in A, then $p \sim q$ means that p and q are Murray-von Neumann equivalent, $p \lesssim q$ means that p is Murray-von Neumann equivalent to a subprojection of q, and $[p]$ denotes the Murray-von Neumann equivalence class of projections containing p. $\widetilde{A}$ will denote the C^*-algebra obtained by joining an identity to A.

We denote the hereditary C^*-subalgebra of A supported by an open projection p by $\text{her}(p)$ and the hereditary C^*-algebra generated by a set D of elements in $M(A)$ by $\text{her}(D)$.

1.1. **Lemma** [20, 2.1]. *Let A be a C^*-algebra and e, f be two projections in A such that $\|f - fef\| \le \beta < 1/4$. Then there is a unitary $u \in \widetilde{A}$ such that $ufu^* \le e$ and $\|u - 1\| \le 6\beta^{1/2}$.*

The following proposition was proved for separable C^*-algebras in [23]. We give a proof for the σ-unital case.

1.2. **Proposition.** *If A is a σ-unital C^*-algebra with an approximate identity consisting of projections, then A has an increasing sequential approximate identity consisting of projections.*

Proof. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate identity of A consisting of projections and h a strictly positive element of A. Then there are $\lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_n \prec \cdots$ in Λ such that $\|e_{\lambda_n}h - h\| < 1/n$. Then $\{e_{\lambda_n}\}$ is a sequential approximate identity of A consisting of projections. Write $e_n = e_{\lambda_n}$. Let $f_1 = e_1$. There is n_1 such that $\|f_1(1-e_{n_1})f_1\|$ is small enough for Lemma 1.1 to apply. Then there is a unitary $u_1 \in \tilde{A}$ such that $\|u_1 - 1\| < \varepsilon_1$ and $u_1 f_1 u_1^* \leq e_{n_1}$. Let $f_2 = u_1^* e_{n_1} u_1$. Then $f_1 \leq f_2$ and $\|f_2 - e_{n_1}\| < 2\varepsilon_1$. By applying the above argument inductively, we can construct the desired approximate identity. □

1.3. **Definition.** An element x in $M(A)$ is said to be *weakly quasidiagonal* if there exist a_i in A such that $a_i a_j^* = a_i^* a_j = a_i a_j = a_j a_i = 0$ for $i \neq j$ and $x = \sum_{i=1}^\infty a_i + a$ for some a in A, where $\sum_{i=1}^\infty a_i$ is a sum converging in the strict topology. The element x is said to be *quasidiagonal* if there exist mutually orthogonal projections p_i in A with $\sum_{i=1}^\infty p_i = 1$ and a_i in $p_i A p_i$ such that $x = \sum_{i=1}^\infty a_i + a$ for some a in A. The element x is said to be *strongly quasidiagonal* if there exist mutually orthogonal projections p_i in A with $\sum_{i=1}^\infty p_i = 1$ and a bounded sequence $\{\lambda_i\}$ such that $x = \sum_{i=1}^\infty \lambda_i p_i + a$ for some a in A.

1.4. *Remarks on* 1.3. (1) By Dini's theorem, it is easy to see that $\sum_{i=1}^n p_i$ converges to the identity of $M(A)$ in the strict topology.

(2) For any bounded sequence $\{\lambda_i\}$, $\sum_{i=1}^n \lambda_i p_i$ converges strictly to an element $\sum_{i=1}^\infty \lambda_i p_i$ in $M(A)$. This follows from the estimate

$$0 \leq b\left[\sum_{i=n}^m |\lambda_i|^2 p_i\right] b^* \leq \|\{\lambda_i\}\|_\infty^2 b\left[\sum_{i=n}^m p_i\right] b^* \quad \text{for any } b \in A \text{ and } n < m.$$

(3) It is trivial that if x is strongly quasidiagonal, then x is quasidiagonal, and thus x is weakly quasidiagonal. But the converses are not true in general since A may be short of projections. For example, if A is the c_0-direct sum of infinitely many copies of a projectionless C^*-algebra, then we can find elements in $M(A)$ that are weakly quasidiagonal but not quasidiagonal. Similarly, we can also obtain examples of quasidiagonal elements that are not strongly quasidiagonal.

(4) The following facts about Definition 1.3 are easily established:

(a) If x in $M(A)_{\text{s.a.}}$ is weakly quasidiagonal or quasidiagonal, then the elements a_i's and a in the sum $x = \sum_{i=1}^\infty a_i + a$ can be chosen from $A_{\text{s.a.}}$.

(b) If x in $M(A)_{\text{s.a.}}$ is strongly quasidiagonal, then numbers λ_i in the sum $x = \sum_{i=1}^\infty \lambda_i p_i + a$ can be chosen to be real and a can be chosen from $A_{\text{s.a.}}$.

In fact, if $x = \sum_{i=1}^{\infty} b_i + b$, then $x = x^*$ implies $x = \sum_{i=1}^{\infty}(b_i + b_i^*)/2 + (b + b^*)/2$. $a_i = (b_i + b_i^*)/2$ and $a = (b + b^*)/2$ are as desired. Hence (a) holds. The proof for (b) is similar.

(5) If A has FS and x is a selfadjoint element in $M(A)$, then x is weakly quasidiagonal if and only if x is strongly quasidiagonal. In fact, for each $i \geq 1$ we can find mutually orthogonal projections p_{ij} and numbers λ_{ij} such that $\|a_i - \sum_{j=1}^{n_i} \lambda_{ij} p_{ij}\| < 1/2^i$. Then $x = \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \lambda_{ij} p_{ij} + a'$, where a' is in A.

1.5. **Definition.** An element x in $M(A)$ is said to be essentially normal if $\pi(x)$ is normal in $M(A)/A$. The essential spectrum of x, denoted by $\sigma_e(x)$, is the spectrum of $\pi(x)$.

We state the following facts [11; 12] as propositions for convenience.

1.6. **Proposition.** *If A has a sequential approximate identity $\{e_n\}$ with $e_n e_m = e_n$ for $n < m$, then every selfadjoint element h in $M(A)$ can be written as*

$$h = \sum_{i=1}^{\infty} (e_{n_i} - e_{n_{i-1}}) h (e_{n_i} - e_{n_{i-1}})$$
$$+ \sum_{i=1}^{+\infty} [(e_{n_{i+1}} - e_{n_i}) h (e_{n_i} - e_{n_{i-1}}) + (e_{n_i} - e_{n_{i-1}}) h (e_{n_{i+1}} - e_{n_i})] + a$$

for some selfadjoint element a in A and $n_i \nearrow \infty$ (an "essential tri-diagonal form").

Proof. Recursively find e_{n_i} such that $\|(1 - e_{n_{i+1}}) h e_{n_i}\| < 1/2^i$, $i = 0, 1, 2, \ldots$ $(e_0 = 0)$. Then set $a = \sum_{i=1}^{\infty} [(1 - e_{n_{i+1}}) h (e_{n_i} - e_{n_{i-1}}) + (e_{n_i} - e_{n_{i-1}}) h (1 - e_{n_{i+1}})]$, where the sum is convergent in norm to an element of A. □

1.7. **Proposition.** *If A is a C^*-algebra and the p_i are mutually orthogonal projections in A such that $\sum_{i=1}^{\infty} p_i$ is in $M(A)$, then $\sum_{i=1}^{\infty} a_i$ is in $M(A)$ for any norm bounded sequence $\{a_i\}$ in A such that $a_i \in p_i A p_i$.*

Proof. The conclusion follows from the estimate

$$\left\| a \left[\sum_{i=n}^{m} a_i \right] \left[\sum_{i=n}^{m} a_i^* \right] a^* \right\| \leq \max \|a_i\|^2 \left\| a \left[\sum_{i=n}^{m} p_i \right] a^* \right\|$$
$$\leq M \left\| a \left[\sum_{i=n}^{m} p_i \right] a^* \right\| \to 0 \quad \text{as } m > n \to \infty$$

for any element a in A, where M is a constant. □

2. K_1-GROUPS, LIFTING OF PROJECTIONS, AND QUASIDIAGONALITY

In [10] L. G. Brown proved that every projection in $M(A)/A$ lifts to a projection in $M(A)$ if A is an AF algebra (or 'less'). Later G. A. Elliott gave a different proof [19] and M.-D. Choi generalized this result by an elementary argument without using K-theory [15]. For σ-unital C^*-algebras with FS, we relate the lifting of projections to the triviality of K_1-groups and the quasidiagonality and prove a more general result by a new method.

2.1. **Examples.** the Calkin algebra has FS and a nontrivial K_1-group $\mathbf{Z}$ (by [7, 9.3.1]). If A is one of Bunce-Deddens algebras (see [5]), then A is a simple, separable C^*-algebra with FS [5; 8, §4] and $K_1(A)$ is nontrivial (see [7, 10.11.4]). These C^*-algebras provide basic examples of non-AF algebras but with FS.

2.2. **Lemma.** *Suppose that A is a C^*-algebra. If every h in $M(A)_+$ satisfying $\pi(h) = \pi(h)^2$ is weakly quasidiagonal then every projection in $M(A)/A$ lifts to a projection in $M(A)$. Consequently, if the general Weyl-von Neumann theorem holds in $M(A)$, then every projection in $M(A)/A$ lifts.*

Proof. If $\overline{q}$ in $M(A)/A$ is a projection and $h_1 \in \pi^{-1}(\overline{q})$, then $h = h_1^* h_1$ is in $\pi^{-1}(\overline{q})$. By hypothesis and Remark 1.4, there exist a_i, a in $A_{\text{s.a.}}$ such that $h = \sum_{i=1}^{\infty} a_i + a$, where $a_i a_j = 0$ if $i \neq j$. Since $\pi(h) = \pi(h)^2$, $h^2 - h \in A_{\text{s.a.}}$. On the other hand, $h^2 - h = \sum_{i=1}^{\infty}(a_i^2 - a_i) + a_0$ for some $a_0 \in A_{\text{s.a.}}$. Thus $b = \sum_{i=1}^{\infty}(a_i^2 - a_i)$ is in $A_{\text{s.a.}}$. Since $\sum_{i=1}^{\infty}(a_i^2 - a_i)$ converges to b in the strict topology and $(a_i^2 - a_i)(a_j^2 - a_j) = 0$ for $i \neq j$, $\varepsilon_i = \|a_i^2 - a_i\| = \|(a_i^2 - a_i)^2\|^{1/2} = \|b(a_i^2 - a_i)\|^{1/2} \to 0$ as $i \to \infty$. Let n be large enough so that $\varepsilon_i < 1/4$ whenever $i \geq n$. Let δ_i be the smaller root of $t^2 - t - \varepsilon_i = 0$. Then $\sigma(a_i) \subset (-\delta_i, \delta_i) \cup (1 - \delta_i, 1 + \delta_i)$. Thus $p_i = \chi_{(\delta_i, 1+\delta_i)}(a_i)$ is in $\text{her}(a_i)$, $\|a_i - p_i\| < \delta_i$ for $i \geq n$, and $p_i p_j = 0$ for $i \neq j$. It follows that $\sum_{i=n}^{\infty} p_i = p$ defines a projection in $M(A)$.

We claim that p is as desired. In fact, since $\delta_i \to 0$, $\sum_{i=n}^{\infty}(a_i - p_i) \in A$. It follows that

$$\begin{aligned}\pi(h - p) &= \pi\left(h - \sum_{i=1}^{\infty} a_i\right) + \pi\left(\sum_{i=1}^{n-1} a_i\right) + \pi\left(\sum_{i=n}^{\infty}(a_i - p_i)\right) \\ &= \pi(a_0) + \pi\left(\sum_{i=n}^{\infty}(a_i - p_i)\right) = 0. \quad \square\end{aligned}$$

2.3. **Corollary.** *If A is a C^*-algebra and the generalized Weyl-von Neumann theorem holds in $L(H_A)$, then $K_1(A) = 0$.*

Proof. The following six-term sequence is exact (see [7, 9.3.1]):

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(M(A \otimes K)) & \longrightarrow & K_0(M(A \otimes K)/A \otimes K) \\ \uparrow & & & & \downarrow \\ K_1(M(A \otimes K)/A \otimes K) & \longleftarrow & K_1(M(A \otimes K)) & \longleftarrow & K_1(A) \end{array}$$

Since $K_0(M(A \otimes K)) = K_1(M(A \otimes K)) = 0$ (see [16, 3.2 or 7, 12.2.1]), $K_1(A) \cong K_0(M(A \otimes K)/A \otimes K)$. On the other hand, it follows from Lemma 2.2 that every projection in $M(A\otimes K)/A\otimes K$ lifts to a projection in $M(A\otimes K)$, and so $K_0(M(A\otimes K))\to K_0(M(A\otimes K)/A\otimes K)$ is onto. Therefore $K_1(A)=0$. □

2.4. **Lemma.** *If B is a unital C^*-algebra, p and q are two projections in B, and $p=\left(\begin{smallmatrix} a & b \\ b^* & c\end{smallmatrix}\right)$ is the decomposition of p with respect to $1=q+(1-q)$, then $\sigma(a)\backslash\{0,1\}=\sigma(1-c)\backslash\{0,1\}$.*

Proof. First of all, it is easy to see that $\left(\begin{smallmatrix} a & b \\ b^* & c\end{smallmatrix}\right)$ is a projection if and only if $a-a^2=bb^*$, $ab=b(1-c)$, $c-c^2=b^*b$, $0\le a\le q$, $0\le c\le 1-q$, $\|a\|\le 1$, and $\|c\|\le 1$. Let $\lambda\notin\sigma(1-c)\cup\{0,1\}$. Then $[1-c-\lambda]^{-1}$ exists in B. Let $f(t)=(t-\lambda)^{-1}$. Then $f(t)$ is continuous on $\sigma(1-c)$. Take a sequence of polynomials $p_n(t)$ defined on $[-2,2]$ such that $p_n(t)\to f(t)$ uniformly on $\sigma(1-c)$. Since $ab=b(1-c)$, it follows that $p_n(a)b=bp_n(1-c)$. It is clear that $p_n(1-c)\to f(1-c)=[1-c-\lambda]^{-1}$ in norm and so the restriction of $p_n(a)$ to p_b converges in the w^*-topology to an element d in $p_bB^{**}p_b$, where p_b is the range projection of b. Hence $db=b[1-c-\lambda]^{-1}$. Using $ab=b(1-c)$ again, we see that

$$(a-\lambda)db=(a-\lambda)b[1-c-\lambda]^{-1}=b[1-c-\lambda][1-c-\lambda]^{-1}=b.$$

Therefore, $a-\lambda$ and d are mutual inverses in $p_bB^{**}p_b$. Let

$$x=d-(1/\lambda)(1-p_b).$$

Then $x\in B^{**}$ and $(a-\lambda)x=p_b+(1-p_b)=1$. It follows that x is the inverse of $(a-\lambda)$ in B^{**}. Thus $(a-\lambda)$ is invertible in B also (see [18, 1.3.10]). Hence $\lambda\notin\sigma(a)$. Therefore, $\sigma(a)\backslash\{0,1\}\subset\sigma(1-c)\backslash\{0,1\}$.

Similarly, if $\lambda\notin\sigma(a)\cup\{0,1\}$, then $(a-\lambda)^{-1}$ exists. Using $b^*a=(1-c)b^*$ and approximating $f(t)=(t-\lambda)^{-1}$ by a sequence of polynomials uniformly on $\sigma(a)$, we get an element $d_1\in p_{b^*}B^{**}p_{b^*}$ such that $b^*(a-\lambda)^{-1}=d_1b^*$. Then $y=d_1-(1/\lambda)(1-p_{b^*})$ is the inverse of $(1-c-\lambda)$ in B^{**}. Then $(1-c-\lambda)$ is invertible in B also. Therefore, $\sigma(1-c)\backslash\{0,1\}\subset\sigma(a)\backslash\{0,1\}$. This completes the proof. □

G. K. Pedersen pointed out an easier proof for Lemma 2.4 as follows: Assume that $0<\lambda<1$. Using the well-known fact that $\sigma(xy)\backslash\{0\}=\sigma(yx)\backslash\{0\}$ for any two elements in a Banach algebra, we see that $\lambda\in\sigma(a)=\sigma(pqp)$ if and only if $1-\lambda\in\sigma(p-pqp)=\sigma(p(1-q)p)$ if and only if $1-\lambda\in\sigma((1-q)p(1-q))=\sigma(c)$.

2.5. **Lemma.** *Assume that A is a C^*-algebra with FS and two projections $\overline{p}$ and $\overline{q}$ in $M(A)/A$ lift to projections in $M(A)$.*

(1) *If $\overline{p}\perp\overline{q}$ lifts to a projection q in $M(A)$, then we can choose a projection $p\perp q$ such that $\pi(p)=\overline{p}$.*

(2) *If $\overline{p}\le\overline{q}$ and $\overline{q}$ lifts to a projection q in $M(A)$, then we can choose a projection p in $M(A)$ such that $p\le q$ and $\pi(p)=\overline{p}$. If $\overline{p}\le\overline{q}$ and $\overline{p}$ lifts to a projection p in $M(A)$, then we can choose a projection q in $M(A)$ such that $p\le q$ and $\pi(q)=\overline{q}$.*

(3) *If every projection in* $M(A)/A$ *lifts, then two commuting projections in* $M(A)/A$ *lift to two commuting projections in* $M(A)$.

Proof. (1) Let p_1 and q be any projections in $M(A)$ with $\pi(p_1) = \overline{p}$ and $\pi(q) = \overline{q}$. $\overline{p} \perp \overline{q}$ implies $\pi(p_1 q) = 0$, i.e., $p_1 q \in A$. Since A has FS, we can find a projection $p_0 \leq p_1$ such that

$$\begin{aligned}\|q(p_1 - p_0)q\| &= \|(p_1 - p_0)q(p_1 - p_0))\| \\ &= \|(p_1 - p_0)(p_1 q p_1)(p_1 - p_0)\| < 1/4.\end{aligned}$$

Let $p_1 - p_0 = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ be the decomposition with respect to $1 = q + (1-q)$. Then $a = q(p_1 - p_0)q$, $c = (1-q)(p_1 - p_0)(1-q)$, and $b = q(p_1 - p_0)(1-q)$. Then $\|a\| < 1/4$ and so $\sigma(a) \subset [0, 1/4]$. It follows that $\sigma(c) \subset \{0\} \cup [3/4, 1]$ by Lemma 2.4. Therefore, $p = \chi_{(1/2,1]}(c)$ is a projection in $(1-q)M(A)(1-q)$, i.e., $p \leq 1 - q$. Since $\overline{p} \perp \overline{q}$,

$$\begin{aligned}\pi(p) &= \pi(\chi_{(1/2,1]}(c)) = \chi_{(1/2,1]}(\pi(c)) \\ &= \chi_{(1/2,1]}((\overline{1} - \overline{q})\overline{p}(\overline{1} - \overline{q})) = \overline{p} \quad \text{(by [18, 1.5.3])}.\end{aligned}$$

Hence p is what we want.

(2) Let p_1 and q be any projections in $M(A)$ with $\pi(p_1) = \overline{p}$ and $\pi(q) = \overline{q}$. Then $\pi(p_1 q) = \overline{p}\,\overline{q} = \overline{p}$, since $\overline{p} \leq \overline{q}$. Then $p_1 q - p_1 \in A$ and so $p_1(1-q)p_1 \in p_1 A p_1$. Since A has FS, we can choose a projection p_0 in $p_1 A p_1$ such that

$$\|(p_1 - p_0)[p_1(1-q)p_1](p_1 - p_0)\| = \|(p_1 - p_0)(1-q)(p_1 - p_0)\| < 1/4.$$

(Just take p_0 as a member of an approximate identity of $p_1 A p_1$ consisting of projections.)

Let $p_1 - p_0 = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ as above. Then

$$\|c\| = \|(1-q)(p_1 - p_0)(1-q)\| = \|(p_1 - p_0)(1-q)(p_1 - p_0)\| < 1/4.$$

It follows that $\sigma(c) \subset [0, 1/4]$, and so $\sigma(a) \subset \{0\} \cup [3/4, 1]$ by Lemma 2.4. Therefore, $g(t) = \chi_{[1/2,1]}(t)$ is continuous on $\sigma(a)$ and so $g(a) = p$ is a projection in $qM(A)q$. We claim that p is as desired. In fact, by [18, 1.5.3],

$$\pi(p) = \pi(f(a)) = f(\pi(a)) = f(\pi(q)\pi(p_1 - p_0)\pi(q)) = f(\overline{q}\,\overline{p}\,\overline{q}) = f(\overline{p}) = \overline{p},$$

since $\pi(p_0) = 0$ and $\overline{p} \leq \overline{q}$. $p \leq q$ is clear. For the second sentence of the conclusion, by applying (1) to $\pi(p)$ and $\overline{1} - \overline{q}$, we can find a projection $q_1 \in M(A)$ such that $q_1 \perp p$ and $\pi(q_1) = \overline{1} - \overline{q}$. Let $q = 1 - q_1$. Then $p \leq q$ and $\pi(q) = \overline{q}$.

(3) Let $\overline{p}$ and $\overline{q}$ be two commuting projections in $M(A)/A$. Then $\overline{r} = \overline{q}\,\overline{p}$ is a projection. By hypothesis, $\overline{p}$ lifts to a projection p in $M(A)$. Since $\overline{r} \leq \overline{p}$, we can find a projection $r \in M(A)$ such that $r \leq p$ and $\pi(r) = \overline{r}$ by (1). Since $\overline{q} - \overline{r} \perp \overline{p}$, (2) applies, so we can find a projection $q_1 \in M(A)$ such that $\pi(q_1) = \overline{q} - \overline{r}$ and $q_1 \perp p$. Let $q = q_1 + r$. Then $pq = qp = r$ and $\pi(q) = \pi(q_1) + \pi(r) = \overline{q}$. □

The following theorem gives characterizations of the liftability of projections.

2.6. **Theorem.** *If A is a σ-unital C^*-algebra with FS, then the following statements are equivalent*:

(a) *Every projection in $M(A)/A$ lifts to a projection in $M(A)$.*
(b) *Every essentially normal element in $M(A)$ with finite essential spectrum is strongly quasidiagonal.*
(c) *Every element h in $M(A)$ with $\pi(h) = \pi(h)^2$ is weakly quasidiagonal.*
(d) *For any projection $\overline{q}$ in $M(A)/A$, there exists a selfadjoint element h in $\pi^{-1}(\overline{q})$ with a "tri-diagonal form" (as in Proposition 1.6)*

$$h = \begin{pmatrix} a_1 & b_1 & & & 0 \\ b_1^* & a_2 & b_2 & & \\ & b_2^* & a_3 & b_3 & \\ & & b_3^* & a_4 & b_4 \\ & 0 & & \ddots & \ddots & \ddots \end{pmatrix}$$

such that $\|b_n\| < 1/2 - \varepsilon$ for some ε with $0 < \varepsilon < 1/2$ and for infinitely many n. Here $a_n = (e_n - e_{n-1})h(e_n - e_{n-1})$ and $b_n = (e_n - e_{n-1})h(e_{n+1} - e_n)$ for some increasing sequential approximate identity $\{e_n\}$ of A consisting of projections.

Proof. (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) follows from Lemma 2.2.

(a) $\Rightarrow$ (b) Let h in $M(A)$ be an essentially normal element with finite essential spectrum, say $\sigma_e(h) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then there exist mutually orthogonal projections $\overline{p}_1, \overline{p}_2, \dots, \overline{p}_n$ in $M(A)/A$ such that $\pi(h) = \sum_{i=1}^n \lambda_i \overline{p}_i$. By applying Lemma 2.5 repeatedly, we find mutually orthogonal projections p_1, $p_2, \dots, p_n \in M(A)$ such that $\pi(p_i) = \overline{p}_i$. It follows that $\pi(h - \sum_{i=1}^n \lambda_i p_i) = 0$. Let $p_{n+1} = 1 - \sum_{i=1}^n p_i$ and $\lambda_{n+1} = 0$. Since A is σ-unital, the algebras $\mathrm{her}(p_i)$ $(1 \le i \le n)$ are all σ-unital (see [11, 3.34]). Since A has FS, every $\mathrm{her}(p_i)$ has an increasing sequential approximate identity consisting of projections by Proposition 1.2. We can write $p_i = \sum_{j=1}^\infty p_{ij}$ $(1 \le i \le n+1)$ for some mutually orthogonal projections p_{ij} in $p_i A p_i$ with the sums converging in the strict topology. Therefore, $\sum_{i=1}^n \lambda_i p_i = \sum_{j=1}^\infty \sum_{i=1}^{n+1} \lambda_{ij} p_{ij}$, where $\lambda_i = \lambda_{i1} = \lambda_{i2} = \cdots = \lambda_{im} = \cdots$ for $1 \le i \le n+1$ and $h = \sum_{j=1}^\infty \sum_{i=1}^{n+1} \lambda_{ij} p_{ij} + a$ for some $a \in A$. It is clear that $\sum p_{ij} = 1$. Thus h is strongly quasidiagonal.

(a) $\Rightarrow$ (d) For any projection $\overline{q} \in M(A)/A$, let q be a projection in $\pi^{-1}(\overline{q})$. Since A is σ-unital with FS, we can write $q = \sum_{i=1}^\infty r_i$ and $1 - q = \sum_{i=1}^\infty r_i'$ for some mutually orthogonal projections r_i, r_i' in A, where the sums converge in the strict topology. Let $a_i = r_i + r_i'$ and $b_i = 0$ for all i. Then q has the desired form.

(d) $\Rightarrow$ (a) Since $\pi(h) = \overline{q}$, $\pi(1 - h) = \overline{1} - \overline{q}$, and so $\pi(h(1-h)) = \overline{0}$. Then $h(1-h) = a$ is an element in A. If n is large enough, we have

$$\|(1 - e_n)a(1 - e_n)\| < \varepsilon - \varepsilon^2.$$

Since

$$(1-e_n)a(1-e_n) = (1-e_n)h(1-e_n) - (1-e_n)h^2(1-e_n)$$
$$= (1-e_n)h(1-e_n) - [(1-e_n)h(1-e_n)]^2 - (1-e_n)he_nh(1-e_n),$$

it follows that

$$\|(1-e_n)h(1-e_n) - [(1-e_n)h(1-e_n)]^2\|$$
$$\leq \|(1-e_n)a(1-e_n)\| + \|(1-e_n)he_nh(1-e_n)\|$$
$$< \varepsilon - \varepsilon^2 + \|(1-e_n)he_n\|^2 = \varepsilon - \varepsilon^2 + \|b_n\|^2$$
$$< \varepsilon - \varepsilon^2 + (1/2-\varepsilon)^2 = 1/4$$

for an appropriate n. Let $h_n = (1-e_n)h(1-e_n)$. Then $\pi(h_n) = \pi(h) = \overline{q}$, since $e_n \in A$. Moreover $\|h_n - h_n^2\| < 1/4$ implies that $\sigma(h_n) \not\ni 1/2$. Let $q_n = \chi_{(1/2,1]}(h_n)$. Then $q_n \in M(A)$ is a projection and $\pi(q_n) = \chi_{(1/2,1]}(\pi(h_n)) = \chi_{(1/2,1]}(\overline{q}) = \overline{q}$. This completes the proof. □

2.7. **Corollary.** *If A is a σ-unital AF algebra, then every essentially normal element in $M(A)$ with finite essential spectrum is strongly quasidiagonal.*

Proof. Since every projection in $M(A)/A$ lifts (see [10]), Theorem 2.6 applies. □

2.8. **Lemma.** *If A is a C^*-algebra and a projection $\overline{p}$ in $M(A)/A$ lifts to a projection p in $M(A)$ such that* her(p) *has an approximate identity consisting of projections, then $\overline{q} \in M(A)/A$ lifts to a projection q in $M(A)$ whenever $\overline{q} \sim \overline{p}$; moreover q can be chosen such that $q \sim p - p_0$ for some projection* $p_0 \in$ her(p).

Proof. Since $\overline{p} \sim \overline{q}$, there exists a partial isometry $\overline{v} \in M(A)/A$ such that $\overline{v}\,\overline{v}^* = \overline{q}$ and $\overline{v}^*\overline{v} = \overline{p}$. Let v be a preimage of $\overline{v}$ in $M(A)$. Since $\pi(vp) = \pi(v)\pi(p) = \overline{v}\,\overline{p} = \overline{v}$, we assume that $vp = v$. Since $\pi(v^*v - p) = \overline{0}$, $a = v^*v - p$ is in pAp. By hypothesis, we can choose a projection $p_0 \leq p$ such that $p_0 \in pAp$ and $\|(p-p_0)(p-v^*v)(p-p_0)\| < 1$. Let $x = (p-p_0)v^*v(p-p_0)$. Then x is invertible in $(p-p_0)M(A)(p-p_0)$. Let y be the inverse of x in $(p-p_0)M(A)(p-p_0)$, i.e., $yx = xy = p - p_0$. Then $y \geq 0$. Let $y^{1/2}v^* = u$. Then $uu^* = y^{1/2}v^*vy^{1/2} = y^{1/2}xy^{1/2} = p - p_0$. It follows that u is a partial isometry in $M(A)$. Clearly, $\pi(x) = \pi(p)^3 = \overline{p}$, since $p_0 \in A$. Consequently, $\pi(y) = \pi(y)\pi(p)^3 = \pi(y)\pi(x) = \pi(yx) = \pi(p-p_0) = \overline{p}$. Hence $\pi(u) = \pi(y^{1/2})\pi(v^*) = \overline{p}\pi(v)^* = \overline{v}^*$, and so $q = u^*u$ is a projection satisfying $q \sim p - p_0$. It is obvious that $\pi(q) = \overline{q}$. □

2.9. *Remark.* The conclusions in Lemmas 2.5 and 2.8 are true in general for the quotient of a C^*-algebra by a σ-unital closed ideal with FS. This generalization is obvious from the proofs.

2.10. **Theorem.** *If A is a C^*-algebra with FS, then the following statements are equivalent:*

(a) $K_1(A) = 0$.

(b) *Every projection $\overline{p}$ in $L(H_A)/K(H_A)$ with $\overline{1} \precsim \overline{p}$ lifts to a projection in $L(H_A)$.*

(c) *Every projection in $L(H_A)/K(H_A)$ lifts to a projection in $L(H_A)$.*

If A is σ-unital, then the following two statements are equivalent to (a)–(c):

(d) *Every essentially normal element in $L(H_A)$ with finite essential spectrum is strongly quasidiagonal.*

(e) *For any projection $\overline{q} \in L(H_A)/K(H_A)$, there exists a selfadjoint element $h \in \pi^{-1}(\overline{q})$ with a "tridiagonal form" (as in Proposition* 1.6)

$$h = \begin{pmatrix} a_1 & b_1 & & & 0 \\ b_1^* & a_2 & b_2 & & \\ & b_2^* & a_3 & b_3 & \\ & & b_3^* & a_4 & b_4 \\ & 0 & & \ddots & \ddots & \ddots \end{pmatrix},$$

such that $\|b_n\| < 1/2 - \varepsilon$ for some ε with $0 < \varepsilon < 1/2$ and infinitely many n. (Notation as in Theorem 2.6.)

Proof. (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follows from Theorem 2.6.

(c) $\Rightarrow$ (a) follows from the proof of Corollary 2.3.

(a) $\Leftrightarrow$ (b) $K_0(M(A\otimes K)/A\otimes K) \cong \{[\overline{p}] : \overline{p}$ is a projection in $M(A\otimes K)/A\otimes K$ and $\overline{1} \precsim \overline{p}\}$ by [16, 3.6]. Since $K_1(A) \cong K_0(M(A\otimes K)/A\otimes K)$ (see [7, 12.2.1]), '$K_1(A) = 0$' implies that $\overline{p} \sim \overline{1}$ whenever $\overline{1} \precsim \overline{p}$. Since $A\otimes K$ has FS [13], Lemma 2.8 applies.

(b) $\Rightarrow$ (c) Let $\overline{q}$ be a projection in $L(H_A)/K(H_A)$. We write the identity of $L(H_A)$ as $1\otimes 1$, where 1 in the first coordinate represents the identity of $M(A)$ and 1 in the second coordinate represents the identity of $\mathrm{L}(H)$. Let v be an isometry with infinite-dimensional corange in $\mathrm{L}(H)$. Then $1\otimes v \in 1\otimes \mathrm{L}(H) \subset L(H_A)$, and $(1\otimes v^*)(1\otimes v) = 1\otimes 1$. Hence

$$\begin{aligned}\overline{1} = \pi(1\otimes 1) &\sim \pi(1\otimes 1 - (1\otimes v)(1\otimes v^*)) \\ &\leq \overline{1} - \pi(1\otimes v)\overline{q}\pi(1\otimes v^*) = \overline{q}_1.\end{aligned}$$

By hypothesis, $\overline{q}_1$ lifts to a projection q_1 in $L(H_A)$. Then $\pi(1\otimes 1 - q_1) = \overline{1} - \overline{q}_1 = \pi(1\otimes v)\overline{q}\pi(1\otimes v^*)$. It follows that $\overline{1} - \overline{q}_1 \sim \overline{q}$. Since A has FS, $\mathrm{her}(1\otimes 1 - q_1)$ has an approximate identity consisting of projections. Therefore Lemma 2.8 applies, and so $\overline{q}$ lifts to a projection in $L(H_A)$. □

2.11. **Corollary.** *If A is a C^*-algebra with FS and $K_1(A) = 0$, then every projection in $M(A)/A$ lifts.*

Proof. First, we identify $M(A)$ with $M(A)\otimes e_{11}$ and A with $A\otimes e_{11}$. Then $M(A)/A$ can be identified with $(M(A)\otimes e_{11})/(A\otimes e_{11})$, which is isomorphic to $(M(A)/A)\otimes e_{11}$.

Let $\overline{q}_1$ be any projection in $(M(A)/A) \otimes e_{11} \subset M(A \otimes K)/A \otimes K$. Since $K_1(A) = 0$ if and only if every projection in $L(H_A)/K(H_A)$ lifts, $\overline{q}_1$ lifts to a projection in $M(A \otimes K)$. Since $\overline{q}_1 \perp \pi(1 \otimes (1 - e_{11}))$ and $A \otimes K$ has FS, Lemma 2.5 applies. So we can find a projection $q_1 \in M(A \otimes K)$ such that $q_1 \perp 1 \otimes (1 - e_{11})$ and $\pi(q_1) = \overline{q}_1$. Hence $q_1 \leq 1 \otimes e_{11}$. This completes the proof. □

2.12. **Corollary.** *If B is any C^*-algebra and A is a closed ideal of B with FS such that $K_1(A) = 0$, then every projection in B/A lifts to a projection in B.*

Proof. Let $\tau : B/A \to M(A)/A$ be the Busby invariant of the extension $0 \to A \to B \to B/A \to 0$ [7, 15.2.1]. Let $C = \{(a, b) \in B/A \oplus M(A) : \pi(b) = \tau(a)\}$. Then C is the pullback of $(B/A, M(A))$ along (τ, π). Thus B is naturally *-isomorphic to C [7, 15.3.2]. Let $\overline{p}$ be any projection in B/A. Then $\tau(\overline{p})$ is a projection in $M(A)/A$. By Corollary 2.11 we can find a projection p in $M(A)$ such that $\pi(p) = \tau(\overline{p})$. Hence $(\overline{p}, p)$ is a projection in C. Let q in B be the projection corresponding to $(\overline{p}, p)$ via the *-isomorphism between C and B. Then q lifts the projection $\overline{p}$. □

3. Characterizations of quasidiagonality

It was proved in [23] that if A is a separable closed ideal of a C^*-algebra B with FS (necessarily A has FS also), then every selfadjoint element of B is quasidiagonal with respect to A. In this section, under the assumption that A is a σ-unital C^*-algebra with FS, we study the quasidiagonality of selfadjoint elements in $M(A)$ ($M(A)$ may not have FS as we will see). We prove that a selfadjoint element of $M(A)$ is strongly quasidiagonal if and only if it can be approximated in norm by selfadjoint elements with finite spectra. As easy consequences, under the assumption that A has FS, the generalized Weyl-von Neumann theorem holds in $M(A)$ if and only if $M(A)$ has FS. If A is stable, a necessary condition for these two equivalent conditions is that $K_1(A)$ be trivial. It will be clear from the proof, that some of the following results are still true for a C^*-algebra B and its σ-unital closed ideal A with FS although we will not state them in such a general form.

Some implications of the following theorem have been proved in [23].

3.1. **Theorem.** *If A is a σ-unital C^*-algebra with FS and h is a selfadjoint element in $M(A)$, then the following statements are equivalent:*

(a) *h is strongly quasidiagonal.*

(b) *For any $\varepsilon > 0$, there exist a bounded real sequence $\{\lambda_i\}$ and mutually orthogonal projections $p_i \in A$ such that $\sum_{i=1}^{\infty} p_i = 1$ and $\|h - \sum_{i=1}^{\infty} \lambda_i p_i\| < \varepsilon$.*

(c) *There exists a sequential approximate identity $\{p_n\}$ of A consisting of projections (not necessarily increasing) such that $\|p_n h - h p_n\| \to 0$.*

(d) *There exists an increasing sequential approximate identity $\{q_n\}$ of A consisting of projections such that $\|q_n h - h q_n\| \to 0$.*

(e) *For any $\varepsilon > 0$, there exist a bounded real sequence $\{\lambda_i\}$ and mutually* *orthogonal projections* $p_i \in A$ *such that* $\sum_{i=1}^{\infty} p_i = 1$ *and* $h = \sum_{i=1}^{\infty} \lambda_i p_i + a$ *for some* a *in* A *with* $\|a\| < \varepsilon$.

(f) *For any* $\varepsilon > 0$, *there exists an element* f_ε *in* $M(A)_{\text{s.a.}}$ *with finite spectrum such that* $\|h - f_\varepsilon\| < \varepsilon$.

Before we go into the proof of the theorem, we state the following corollary.

3.2. **Corollary.** *Suppose that* A *is a* σ*-unital* C^**-algebra with FS. Then the generalized Weyl-von Neumann theorem holds in* $M(A)$ *if and only if* $M(A)$ *has FS. Any of these two equivalent conditions implies that every projection in* $M(A)/A$ *lifts to a projection in* $M(A)$. *If, furthermore,* A *is stable, then '*$K_1(A) = 0$*' is a necessary condition in order that the generalized Weyl-von Neumann theorem holds in* $M(A)$.

Proof. The second conclusion follows from Lemma 2.2. The first conclusion follows from the equivalence of (a) and (f) in Theorem 3.1. □

3.3. *Remark.* An easy consequence of Theorem 3.1(b) is that if A is σ-unital with FS, then the set of elements of the form $\sum_{i=1}^{\infty} \lambda_i p_i$ for some $\{\lambda_i\}$ in l^∞ and a sequence of mutually orthogonal projections $\{p_i\}$ in A with $\sum_{i=1}^{\infty} p_i = 1$ is norm dense in $M(A)_{\text{s.a.}}$ if and only if the generalized Weyl-von Neumann theorem holds in $M(A)$. Notice that $h - \sum_{i=1}^{\infty} \lambda_i p_i$ in Theorem 3.1(b) is not necessarily in A. Hence (b) and (e) seem to state two different things, but they are actually equivalent.

Now we prove the theorem. We will prove the equivalences in the following pattern: (c) $\Leftrightarrow$ (d) in 3.4; (d) $\Rightarrow$ (*e*) $\Rightarrow$ (b) in 3.5; (b) $\Rightarrow$ (c) in 3.6; (e) $\Rightarrow$ (a) $\Rightarrow$ (d) in 3.7; (b) $\Leftrightarrow$ (f) in 3.8.

3.4. *Proof of* (c) $\Leftrightarrow$ (d). (c) $\Leftarrow$ (d) is trivial.

(c) $\Rightarrow$ (d) We may assume that $\|h\| = 1$. Let $\varepsilon_n \searrow 0$ with $(\varepsilon_1/48)^2 < 1/4$. Let $q_1 = p_1$. Since $p_m \to 1$ in the strict topology and $\|p_m h - h p_m\| \to 0$, there is $m_1 > 1$ such that $\|q_1(1 - p_{m_1}) q_1\| < (\varepsilon_1/48)^2 < 1/4$ and $\|p_{m_1} h - h p_{m_1}\| < \varepsilon_1/2$. By Lemma 1.1, there is a unitary $u_1 \in \widetilde{A}$ such that $\|u_1 - 1\| < \varepsilon_1/8$ and $u_1 q_1 u_1^* \le p_{m_1}$. Let $q_2 = u_1^* p_{m_1} u_1$. Then $q_1 \le q_2$ and

$$\begin{aligned} \|q_2 h - h q_2\| &\le \|p_{m_1} h - h p_{m_1}\| + \|(q_2 - p_{m_1}) h - h(q_2 - p_{m_1})\| \\ &< \varepsilon_1/2 + 2\|q_2 - p_{m_1}\| \\ &= \varepsilon_1/2 + 2\|p_{m_1}(u_1 - 1) - (u_1 - 1) p_{m_1}\| \\ &< \varepsilon_1/2 + 4\|u_1 - 1\| < \varepsilon_1. \end{aligned}$$

Repeating this argument, we get

(1) $\|p_{m_i} h - h p_{m_i}\| < \varepsilon_i/2$ for some $m_i \nearrow \infty$;

(2) u_i is unitary in $\widetilde{A}$ with $\|u_i - 1\| < \varepsilon_i/8$;

(3) $q_{i+1} = u_i^* p_{m_i} u_i$, $q_{i-1} \le q_i \le q_{i+1}$, and $\|q_i h - h q_i\| < \varepsilon_i$ for $i = 0, 1, 2, \ldots$. (Here $m_0 = 1$ and $u_0 = 1$.)

It remains to show that $\{q_i\}$ is an approximate identity of A. Let b be any element of A. Then

$$\begin{aligned}\|(1-q_{i+1})b\| &= \|u_i^*(1-p_{m_i})u_ib\| \\ &\le \|(1-p_{m_i})(u_i-1)b\| + \|(1-p_{m_i})b\| \\ &\le \|u_i-1\|\|b\| + \|(1-p_{m_i})b\| \to 0,\end{aligned}$$

since $\|u_i-1\|\to 0$ and $\{p_{m_i}\}$ is an approximate identity of A. ((c) $\Leftrightarrow$ (d) is true for any element h in $M(A)$, not necessarily selfadjoint.) □

3.5. *Proof of* (d) $\Rightarrow$ (e) $\Rightarrow$ (b). (e) $\Rightarrow$ (b) is trivial.

(d) $\Rightarrow$ (e) By a minor modification of the arguments of [22] we recursively obtain a sequence $n_i \nearrow \infty$ such that $\|(q_{n_i}-q_{n_{i-1}})h(1-q_{n_i})\| < \varepsilon/2^{i+2}$ for $i\ge 1$. Then we write h (in A^{**}) as

$$\begin{aligned}h &= \sum_{i=1}^{\infty}(q_{n_i}-q_{n_{i-1}})h(q_{n_i}-q_{n_{i-1}}) \\ &\quad + \sum_{i=1}^{\infty}[(1-q_{n_i})h(q_{n_i}-q_{n_{i-1}}) + (q_{n_i}-q_{n_{i-1}})h(1-q_{n_i})].\end{aligned}$$

It is easy to check that the second sum above is convergent in norm to an element $a'\in A$ with $\|a'\|\le 2\varepsilon\sum_{i=1}^{\infty}1/2^{i+2} < \varepsilon/2$. Thus

$$h-a' = \sum_{i=1}^{\infty}(q_{n_i}-q_{n_{i-1}})h(q_{n_i}-q_{n_{i-1}}) \in M(A).$$

Since A has FS, the algebra $A_i = \mathrm{her}(q_{n_i}-q_{n_{i-1}})$ has FS. We can find a selfadjoint element $y_i\in A_i$ with finite spectrum $\{t_{ij} : 1\le j\le r_i\}$ such that $\|x_i-y_i\| < \varepsilon/2^{i+1}$, where $x_i = (q_{n_i}-q_{n_{i-1}})h(q_{n_i}-q_{n_{i-1}})$. Consequently,

$$\left\|h-\sum_{i=1}^{\infty}y_i\right\| \le \left\|h-\sum_{i=1}^{\infty}x_i\right\| + \left\|\sum_{i=1}^{\infty}(x_i-y_i)\right\| \le \|a'\| + \sum_{i=1}^{\infty}\varepsilon/2^{i+1} < \varepsilon.$$

It is clear that the sum $\sum_{i=1}^{\infty}y_i$ converges to an element in $M(A)$ in the strict topology and $\sum_{i=1}^{\infty}(x_i-y_i)$ converges in norm to an element in A.

By operator calculus, we write $y_i = \sum_{j=1}^{r_i}t_{ij}p_{ij}$ for some mutually orthogonal projections $p_{ij}\in A_i$ $(j=1,2,\dots,r_i)$ such that $\sum_{j=1}^{r_i}p_{ij} = q_{n_i}-q_{n_{i-1}}$ for each $i\ge 1$. Then $\sum_{i=1}^{\infty}\sum_{j=1}^{r_i}p_{ij} = 1$ since $\{q_{n_i}\}$ is an approximate identity of A. Therefore, $h = \sum_{i=1}^{\infty}\sum_{j=1}^{r_i}t_{ij}p_{ij} + a$, where $a = a' + \sum_{i=1}^{\infty}(x_i-y_i)\in A$ and $\|a\| < \varepsilon$. □

3.6. *Proof of* (b) $\Rightarrow$ (c). Let $\varepsilon_n \searrow 0$. By hypothesis, for each n we can find a bounded real sequence $\{t_i(n)\}_{i=1}^{\infty}$ and a sequence of mutually orthogonal projections $\{p_i(n)\}\subset A$ such that $\sum_{i=1}^{\infty}p_i(n) = 1$ and $\|h-\sum_{i=1}^{\infty}t_i(n)p_i(n)\| < \varepsilon_n$. If $q_m(n) = \sum_{i=1}^{m}p_i(n)$, then $\{q_m(n)\}_{m=1}^{\infty}$ is an approximate identity of A

for each n. Let $\{e_n\}$ be a fixed increasing approximate identity of A consisting of projections. (The existence of such an approximate identity is guaranteed by Proposition 1.2). Using Lemma 1.1 repeatedly, we get

$$m_1 > 1,\ u_1 \in U(\widetilde{A}) \quad \text{with } \|u_1 - 1\| < 1/2 \text{ and } u_1 e_1 u_1^* \leq q_{m_1}(1);$$

$$m_2 > m_1,\ u_2 \in U(\widetilde{A}) \quad \text{with } \|u_2 - 1\| < 1/2^2 \text{ and } u_2 e_2 u_2^* \leq q_{m_2}(2);$$

$$\cdots\cdots$$

$$m_i > m_{i-1},\ u_i \in U(\widetilde{A}) \quad \text{with } \|u_i - 1\| < 1/2^i \text{ and } u_i e_i u_i^* \leq q_{m_i}(i);$$

$$\cdots\cdots$$

Let $p_n = q_{m_n}(n)$ for each n. We claim that $\{p_n\}$ is an approximate identity of A (not necessarily increasing) and $\|p_n h - h p_n\| \to 0$. In fact, for any element $a \in A$ we have $0 \leq a^*(1 - p_n)a \leq a^*(1 - u_n e_n u_n^*)a$. Then

$$\begin{aligned}
\|a^*(1 - p_n)a\| &\leq \|a^*(1 - u_n e_n u_n^*)a\| \\
&\leq \|a^*(1 - e_n)a\| + \|a^*(e_n - u_n e_n u_n^*)a\| \\
&\leq \|a^*(1 - e_n)a\| + \|a\|^2 \|e_n(u_n - 1) - (u_n - 1)e_n\| \\
&\leq \|a^*(1 - e_n)a\| + 2\|a\|^2 \|u_n - 1\| \to 0.
\end{aligned}$$

Hence $\{p_n\}$ is an approximate identity of A. On the other hand,

$$\begin{aligned}
\|p_n h - h p_n\| &= \left\| p_n \left[h - \sum_{i=1}^{\infty} t_i(n) p_i(n) \right] - \left[h - \sum_{i=1}^{\infty} t_i(n) p_i(n) \right] p_n \right\| \\
&\leq 2 \left\| h - \sum_{i=1}^{\infty} t_i(n) p_i(n) \right\| < 2\varepsilon_n \to 0,
\end{aligned}$$

since p_n commutes with $\sum_{i=1}^{\infty} t_i(n) p_i(n)$. □

3.7. *Proof of* (e) ⇒ (a) ⇒ (d). (e) ⇒ (a) is trivial.

(a) ⇒ (d) Let $h = \sum_{i=1}^{\infty} \lambda_i p_i + a$, where $\sum_{i=1}^{\infty} p_i = 1$ and the p_i's are mutually orthogonal projections of A, and $a \in A$. Let $q_n = \sum_{i=1}^{n} p_i$ for each $n \geq 1$. Then $\{q_n\}$ is an approximate identity of A satisfying

$$\begin{aligned}
\|q_n h - h q_n\| &= \left\| q_n \left(h - \sum_{i=1}^{\infty} \lambda_i p_i \right) - \left(h - \sum_{i=1}^{\infty} \lambda_i p_i \right) q_n \right\| = \|q_n a - a q_n\| \\
&= \max\{\|q_n a(1 - q_n)\|, \|(1 - q_n) a q_n\|\} \leq \|q_n a(1 - q_n)\| \to 0,
\end{aligned}$$

since q_n commutes with $\sum_{i=1}^{\infty} \lambda_i p_i$. □

3.8. *Proof of* (b) ⇔ (f). (f) ⇒ (b) By hypothesis, there is an element $f_\varepsilon \in M(A)_{\text{s.a.}}$ with finite spectrum $\{t_1, t_2, \ldots, t_n\}$ such that $\|h - f_\varepsilon\| < \varepsilon$. By operator calculus, there are mutually orthogonal projections $e_1, e_2, \ldots, e_n$ in $M(A)$ such that $f_\varepsilon = \sum_{i=1}^{n} t_i e_i$. Since A is σ-unital, algebras $\text{her}(e_i)$ $(1 \leq i \leq n)$ are all σ-unital (see [11, 3.34]). Since A has FS, by Proposition 1.2 there are mutually orthogonal projections $p_{ij} \in A$ $(1 \le j)$ such that $\sum_{j=1}^{\infty} p_{ij} = e_i$ for each i. Rename $\{p_{ij}\}$ by $\{p_j\}$ and $\{\lambda_i\}$ by $\{\mu_j\}$. Then $f_\varepsilon = \sum_{j=1}^{\infty} \mu_j p_j$ as desired.

(b) ⇒ (f) By hypothesis, for any $\varepsilon > 0$ there exist a bounded real sequence $\{\lambda_i\}$ and mutually orthogonal projections $p_i \in A$ with $\sum_{i=1}^{\infty} p_i = 1$ such that $\|h - \sum_{i=1}^{\infty} \lambda_i p_i\| < \varepsilon/2$. Let $\mu = \inf_i\{\lambda_i\}$ and $\mu' = \sup_i\{\lambda_i\}$. Choose n large enough such that $(\mu' - \mu)/n < \varepsilon/2$. Take a subdivision of the interval $[\mu, \mu']$ with n subintervals:

$$\mu = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_{n-1} < \mu_n = \mu'$$

such that $\mu_i - \mu_{i-1} = (\mu' - \mu)/n$ for $i = 1, 2, \dots, n$. Let

$$e_i = \sum_{\{n : \mu_{i-1} < \lambda_n \le \mu_i\}} p_n$$

for each $i = 1, 2, \dots, n$. Then $e_i \in M(A)$ by Proposition 1.7. Since the $\{e_i\}$ are mutually orthogonal, $\sum_{i=1}^{n} \mu_{i-1} e_i$ has finite spectrum. Therefore,

$$\left\| h - \sum_{i=1}^{n} \mu_{i-1} e_i \right\| \le \left\| h - \sum_{j=1}^{\infty} \lambda_j p_j \right\| + \left\| \sum_{j=1}^{\infty} \lambda_j p_j - \sum_{i=1}^{n} \mu_{i-1} e_i \right\| < \varepsilon. \quad \square$$

We now turn to a brief view of the quasidiagonality of general elements (not necessarily selfadjoint) in $M(A)$.

3.9. **Proposition.** *If A is a C^*-algebra with an increasing sequential approximate identity consisting of projections and x is any element in $M(A)$, then the following statements are equivalent*:

(a) *x is quasidiagonal.*

(b) *For any $\varepsilon > 0$, there exist mutually orthogonal projections p_i in A such that $\sum_{i=1}^{\infty} p_i = 1$ and $\|x - \sum_{i=1}^{\infty} p_i x p_i\| < \varepsilon$.*

(c) *There exist a sequential approximate identity $\{p_n\}$ of A consisting of projections (not necessarily increasing) such that $\|p_n x - x p_n\| \to 0$.*

(d) *There exists an increasing sequential approximate identity $\{q_n\}$ of A consisting of projections such that $\|q_n x - x q_n\| \to 0$.*

(e) *For any $\varepsilon > 0$, there exist mutually orthogonal projections p_i in A such that $\sum_{i=1}^{\infty} p_i = 1$ and $x = \sum_{i=1}^{\infty} p_i x p_i + a$ for some a in A with $\|a\| < \varepsilon$.*

Proof. We leave it to the reader to check that a minor modification in the proof of Theorem 3.1 will yield this proposition. □

3.10. **Proposition.** *If A is a σ-unital C^*-algebra with FS, then the following two statements are equivalent*:

(f) *The set of normal elements of $M(A)$ with finite spectra is norm dense in the set of all normal elements of $M(A)$.*

(g) *For any normal element x in $M(A)$ and any $\varepsilon > 0$ there exist mutually orthogonal projections p_i in A and a bounded complex sequence $\{\lambda_i\}$ such that $\sum_{i=1}^{\infty} p_i = 1$ and $\|x - \sum_{i=1}^{\infty} \lambda_i p_i\| < \varepsilon$.*

Moreover the above two statements imply that every normal element in $M(A)$ is strongly quasidiagonal.

Proof. The proof is left to the reader.

4. Interpolation by multiplier projections

If A is a C^*-algebra, we say that the interpolation by multiplier projections holds if: Whenever two closed projections p_1 and p_2 in A^{**} are mutually orthogonal, there exists a projection r in $M(A)$ such that $p_1 \leq r \leq 1 - p_2$. Equivalently, if a closed projection p_1 and an open projection p in A^{**} satisfy $p_1 \leq p$, then there exists a projection r in $M(A)$ such that $p_1 \leq r \leq p$.

Recently L. G. Brown [12] proved that under the assumption that A is a σ-unital C^*-algebra with FS, then $M(A)$ has FS if and only if interpolation by multiplier projections holds. Consequently using Theorem 3.1, we obtain that the generalized Weyl-von Neumann theorem holds in $M(A)$ if and only if interpolation by multiplier projections holds.

In this section we will prove five conditions equivalent to interpolation by multiplier projections. Thus these are equivalent to the six conditions in Theorem 3.1. We are hoping that these conditions give various possible ways to see when the generalized Weyl-von Neumann theorem holds in $M(A)$.

The following lemma will be useful later, which can be regarded as a generalization of [21, 3.2] from separable AF algebras to σ-unital C^*-algebras with FS. The proof borrows ideas from [20].

4.1. **Lemma.** *Let A be a σ-unital C^*-algebra with FS and B a σ-unital hereditary C^*-subalgebra of A.*

(a) *For any increasing sequential approximate identity $\{r_n\}$ of B consisting of projections, there exists an increasing sequential approximate identity $\{p_n\}$ of A consisting of projections such that $r_n \leq p_n$ for each n.*

(b) *If B is an ideal of A, for any given increasing sequential approximate identity $\{p_n\}$ of A consisting of projections, there is an increasing sequential approximate identity $\{r_m\}$ of B consisting of projections and $n_m \nearrow \infty$ such that $r_m \leq p_{n_m}$ for each m.*

Proof. (a) Let $\{q_n\}$ be any sequential increasing approximate identity of A consisting of projections, and let $\varepsilon > 0$. Since $\|(1-q_n)r_1\| \to 0$, we can choose m_1 large enough so that $\|r_1(1-q_{m_1})r_1\|$ is small. Then by Lemma 1.1 there exists a unitary element u_1 in $U(\widetilde{A})$ such that

$$\|u_1 - 1\| < \varepsilon/2 \quad \text{and} \quad u_1 r_1 u_1^* \leq q_{m_1}.$$

Then

$$r_1 \leq u_1^* q_{m_1} u_1 \leq u_1^* q_n u_1 \quad \text{if } n > m_1.$$

Choose $m_2 > m_1$ such that $\|(r_2 - r_1)[(1 - r_1) - (u_1^* q_{m_2} u_1 - r_1)](r_2 - r_1)\|$ is small enough. Then we can use Lemma 1.1 again to get a unitary u_2' in

$U((1-r_1)\widetilde{A}(1-r_1))$ such that

$$\|u_2' - (1-r_1)\| < \varepsilon/2^2 \quad \text{and} \quad u_2'(r_2 - r_1)u_2'^* \le u_1^* q_{m_2} u_1 - r_1.$$

Let $u_2 = u_2' + r_1$. Then $u_2 r_1 = r_1 u_2 = r_1$, $\|u_2 - 1\| < \varepsilon/2^2$, and $r_2 \le u_2^* u_1^* q_{m_2} u_1 u_2 \le u_2^* u_1^* q_n u_1 u_2$ if $n > m_2$. Also $r_1 = u_2^* r_1 u_2 \le u_2^* u_1^* q_{m_1} u_1 u_2 \le u_2^* u_1^* q_{m_2} u_1 u_2$.

By induction, we can find a sequence $m_n \nearrow \infty$ and a sequence of unitaries $\{u_n\}$ in $U(\widetilde{A})$ such that

(1) $\|u_n - 1\| < \varepsilon/2^n$;

(2) $u_n r_i = r_i u_n = r_i$, $i = 1, 2, \dots, n-1$;

(3) $r_i \le u_n^* u_{n-1}^* \cdots u_2^* u_1^* q_{m_i} u_1 u_2 \cdots u_{n-1} u_n \le u_n^* u_{n-1}^* \cdots u_2^* u_1^* q_{m_n} u_1 u_2 \cdots u_{n-1} u_n$ for $i = 1, 2, \dots, n$.

Define $u_\varepsilon = \prod_{i=1}^\infty u_i$. Then u_ε is well defined in $\widetilde{A}$, since

$$\begin{aligned}
&\|u_1 u_2 \cdots u_m - u_1 u_2 \cdots u_n\| \\
&\quad = \|u_1 u_2 \cdots u_{m-1}(u_m - 1) + u_1 \cdots u_{m-2}(u_{m-1} - 1) \\
&\qquad\qquad + \cdots + u_1 \cdots u_n (u_{n+1} - 1)\| \\
&\quad \le \sum_{j=n+1}^{m} \|u_j - 1\| < \sum_{j=n+1}^{m} \varepsilon/2^j \to 0
\end{aligned}$$

as $n \nearrow \infty$. It is easy to see that $\|u_\varepsilon - 1\| < \varepsilon$ since

$$\|u_\varepsilon - 1\| = \lim_{n\to\infty} \|u_1 u_2 \cdots u_n - 1\| \le \lim_{n\to\infty} \sum_{j=1}^{n} \|u_j - 1\| < \sum_{j=1}^{\infty} \varepsilon/2^j = \varepsilon.$$

Let $p_n = u_\varepsilon^* q_{m_n} u_\varepsilon$. Then $\{p_n\}$ is as desired.

(b) If B is an ideal of A, we take any increasing approximate identity $\{r_m'\}$ of B consisting of projections. Then by repeating the arguments in the proof of (a), we can get a unitary $u_\varepsilon \in U(\widetilde{A})$ and some $n_m \nearrow \infty$ such that $r_m' \le u_\varepsilon^* p_{n_m} u_\varepsilon$. Let $r_m = u_\varepsilon r_m' u_\varepsilon^*$ $(m \ge 1)$. □

We give five conditions equivalent to the interpolation by multiplier projections in the following theorem. First we recall a result on the relative positions of two hereditary C^*-subalgebras. Assume that p and q are two open projections. We say that $\mathrm{her}(p)$ and $\mathrm{her}(q)$ are q-commuting if $pq = qp$. It was proved [2] that if p and q commute, then $pq = r$ is an open projection as well. It is easy to see that $\mathrm{her}(r) = \mathrm{her}(p) \cap \mathrm{her}(q)$.

4.2. **Theorem.** *If A is a σ-unital C^*-algebra with FS, then the following conditions are equivalent*:

(a) *Interpolation by multiplier projections holds.*

(b) *If p and q are commuting open projections such that* $\mathrm{her}(p)$, $\mathrm{her}(q)$, *and* $\mathrm{her}(r)$ *are all σ-unital with $p \vee q = 1$, then $r = e + f$ for some projection* *e in $M(B)\cap M(D)$ and some projection f in $M(C)\cap M(D)$, where $pq=r$, $B=\mathrm{her}(p)$, $C=\mathrm{her}(q)$, and $D=\mathrm{her}(r)$.*

(c) *Under the assumptions in* (b), *there is a sequence of projections $\{r_n\}$ in D such that $\|b(1-r_n)c\|\to 0$ for any $b\in B$ and $c\in C$ (consequently $\{r_n\}$ is an approximate identity of D).*

(c$'$) *Same as* (c) *expect that $\{r_n\}$ is required to be increasing.*

(d) *Under the assumptions in* (b), *there exist sequential increasing approximate identities $\{p_n\}$ and $\{q_n\}$ of B and C, respectively, consisting of projections such that $p_nq_n=q_np_n$ (and so $\{p_nq_n\}$ is an increasing approximate identity of D).*

(e) *Under the assumptions of* (b), *there are sequential approximate identities $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$ of B, C, and D respectively consisting of projections such that*

(1) $p_n\ge r_n\le q_n$ *for* $n=1,2,\dots$,

(2) $\|p_{n-1}(r_n-r_{n-1})q_{n-1}(r_n-r_{n-1})\|=\mu_n<\delta_n^3$ *for* $n\ge 2$,

where $\{\delta_n\}$ is any sequence of positive numbers such that $\sum_{n=1}^{\infty}\delta_n<\infty$.

The proof will be given after the following remark.

4.3. *Remark.* In statement (e), condition (1) can always be achieved by Lemma 4.1. It can be done by fixing $\{r_n\}\subset D$, choosing $p_n\in B$ with $r_n\le p_n$, and choosing $q_n\in C$ with $r_n\le q_n$, respectively, for each n. So condition (2) is the key point. Let us take a look at the condition (2) to see what it means: If we write

$$p_{n-1}-r_{n-1}=\begin{pmatrix} a_n & b_n\\ b_n^* & c_n\end{pmatrix}\quad\text{and}\quad q_{n-1}-r_{n-1}=\begin{pmatrix} a_n' & b_n'\\ b_n' & c_n'\end{pmatrix}$$

with respect to $1-r_{n-1}=(r_n-r_{n-1})+(1-r_n)$, then $a_n=(r_n-r_{n-1})p_{n-1}(r_n-r_{n-1})$ and $a_n'=(r_n-r_{n-1})q_{n-1}(r_n-r_{n-1})$. Therefore,

$$\begin{aligned}\|a_na_n'\|&=\|(r_n-r_{n-1})p_{n-1}(r_n-r_{n-1})q_{n-1}(r_n-r_{n-1})\|\\ &\le\|p_{n-1}(r_n-r_{n-1})q_{n-1}(r_n-r_{n-1})\|\\ &=\|(r_n-r_{n-1})q_{n-1}(r_n-r_{n-1})p_{n-1}(r_n-r_{n-1})q_{n-1}(r_n-r_{n-1})\|^{1/2}\\ &\le\|a_na_n'\|^{1/2}.\end{aligned}$$

This means that $\|a_na_n'\|$ is small if and only if $\|p_{n-1}(r_n-r_{n-1})q_{n-1}(r_n-r_{n-1})\|$ is small. The condition (e) is equivalent to (d), but the construction of $\{p_n\}$ and $\{q_n\}$ in (e) should be easier. This is the motivation to prove such an equivalence condition.

Now we turn to the proof of Theorem 4.2. We arrange the proof in the following pattern: (a) $\Leftrightarrow$ (b) in 4.4; (b) $\Rightarrow$ (c) $\Rightarrow$ (c$'$) in 4.5; (c$'$) $\Rightarrow$ (b) in 4.6; (b) $\Rightarrow$ (d) $\Rightarrow$ (e) in 4.7; (e) $\Rightarrow$ (b) in 4.8.

4.4. *Proof of* (a) $\Leftrightarrow$ (b). (a) $\Rightarrow$ (b). Let $p_1=1-p$ and $p_2=1-q$. Then p_1 and p_2 are closed projections in A^{**}, $p=1-p_1$, and $q=1-p_2$. Since

$pq = qp = r$, it follows that $p_1 p_2 = 0$. By (a) there is a projection $s \in M(A)$ such that $p_1 \le s \le 1 - p_2$.

Let $e = sr$ and $f = (1-s)r$. Then $e = sp$ and $e + f = r$. For any b in B, $pb = bp = b$ and $be = pbpe = pbep = pbsp \in pAp$ since $s \in M(A)$. On the other hand, $be = b(ps) = (bp)s = bs \in A$. Then $be \in pAp \cap A \subset B$, and so $e \in M(B)$. Clearly, $e \in M(D)$. Similarly, $cf = qc(1-s)q \in qAq$ and $cf = c(1-s) \in A$ for any $c \in C$. Hence $f \in M(C) \cap M(D)$.

(b) $\Rightarrow$ (a) Let p_1 and p_2 be any two closed projections in A^{**} such that $p_1 p_2 = 0$. Then $(1-p_1)(1-p_2) = (1-p_2)(1-p_1) = r$ and $p_1 + p_2 + r = 1$. Hence $B = \mathrm{her}(1-p_1)$ and $C = \mathrm{her}(1-p_2)$ are q-commuting. It is obvious that $(1-p_1) \vee (1-p_2) = 1$, i.e., $A = \mathrm{her}(B \cup C)$. By [11, 3.30], there are q-commuting hereditary C^*-subalgebras B' of B and C' of C such that $b \in B'$ and $c \in C'$, and B', C' and $B' \cap C'$ are all σ-unital. Also $\mathrm{her}(B' \cup C') = A$. If p_1' and p_2' are the closed projections corresponding to B' and C' respectively, then $p_i \le p_i'$ $(i = 1, 2)$ and $p_1' p_2' = 0$. Changing notation, we may assume that B, C, and $D = B \cap C$ are all σ-unital.

Let $r = e + f$ for some e in $M(B) \cap M(D)$ and some f in $M(C) \cap M(D)$. Define $s = p_1 + e$; then $1 - s = 1 - p_1 - e = (p_2 + r) - e = p_2 + f$, since $p_1 + p_2 + r = 1$. For any b in B, $bs = b(1-p_1)s = be \in B$, since $e \in M(B)$. Thus $s \in M(B)$. For any c in C, $c(1-s) = c(p_2 + f) = c(1-p_2)(p_2 + f) = cf \in C$, since $f \in M(C)$. Then $s \in M(C)$. Thus $cs = c - cf \in C$. So $s \in M(B) \cap M(C) \subset M(A)$. The inclusion follows from $\mathrm{her}(B \cup C) = A$. □

4.5. *Proof of* (b) $\Rightarrow$ (c) $\Rightarrow$ (c$'$). Since e in $M(B)$ is a projection and A has FS, there exist projections $e_n \in D$ such that $e_n \nearrow e$ and $b(e - e_n) \to 0$ in norm for any b in B. Similarly, there exist projections f_n in D such that $f_n \nearrow f$ and $(f - f_n)c \to 0$ in norm for any c in C. Let $r_n = e_n + f_n$. Then $r_n \in D$. For any b in B and c in C, we have

$$\|b(1-r_n)c\| = \|b(r-r_n)c\| \le \|b(e-e_n)\|\,\|c\| + \|b\|\,\|(f-f_n)c\| \to 0$$

since $bc = brc$ (see [11, 3.29]). It is obvious that $\{r_n\}$ is an approximate identity of D. Hence, (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (c$'$) Let $\{r_n\}$ be any sequence of projections (not necessarily increasing) such that the condition (c) holds. Choose $n_1 > 1$ such that $\|(1-r_{n_1})r_1\|$ is small enough so that Lemma 1.1 applies to give a unitary u_1 in $U(\tilde{D})$ with the properties $u_1^* r_1 u_1 \le r_{n_1}$ and $\|u_1 - r\| < \varepsilon_1$. Let $r_1' = r_1$ and $r_2' = u_1 r_{n_1} u_1^*$ Then $r_1' \le r_2' \in D$.

Choose $n_2 > n_1$ such that $\|(1-r_{n_2})r_2'\|$ is small. Then Lemma 1.1 applies to obtain a unitary $u_2 \in U(\tilde{D})$ with $u_2^* r_2' u_2 \le r_{n_2}$ and $\|u_2 - r\| < \varepsilon_2$. Let $r_3' = u_2 r_{n_2} u_2^*$. Then $r_2' \le r_3'$.

Proceeding in this way, we get a sequence $n_m \nearrow \infty$ and a sequence of unitaries $\{u_n\} \subset U(\tilde{D})$ such that $\|u_m - r\| < \varepsilon_m$ and $r_m' \le r_{m+1}' = u_m r_{n_m} u_m^* \in D$. By construction, $\{r_m'\}$ is increasing. It remains to show that $\|b(1-r_m')c\| \to$

0 for any b in B and c in C. In fact,

$$\begin{aligned}\|b(1-r'_{m+1})c\| &= \|b(r-r'_{m+1})c\| \\ &\le \|b(u_m-r)(r-r_{n_m})u_m^*c\| + \|b(r-r_{n_m})(u_m^*-r)c\| \\ &\quad + \|b(r-r_{n_m})c\| \\ &< 2\|b\|\,\|c\|\varepsilon_m + \|b(r-r_{n_m})c\| \to 0. \quad \square\end{aligned}$$

4.6. *Proof of* (c′) ⇒ (b). Let b and c be strictly positive elements of norm 1 in B and C, respectively, and $\{r_n\}$ an increasing approximate identity of D as in (c). Let $\{\delta_n\}$ be any sequence of positive numbers with $\sum_{n=1}^{\infty}\delta_n^{1/2} < \infty$ and $\{\varepsilon_n\}$ a sequence of positive numbers such that $\varepsilon_n \le \delta_n/6n^2$ for each n.

Since $\|b(1-r_n)c\| \to 0$, we can choose $n_1 > 1$ such that

$$\|b(1-r_{n_1})c\| = \|b(r-r_{n_1})c\| \le \varepsilon_2.$$

Since $\|b(1-r_n)c\| \to 0$, we can choose $n_2 > n_1$ such that $\|b(r-r_{n_2})c\| \le \varepsilon_3$. It follows that

$$\|b(r_{n_2}-r_{n_1})c\| \le \|b(r-r_{n_2})c\| + \|b(r-r_{n_1})c\| \le \varepsilon_3 + \varepsilon_2 \le 2\varepsilon_2.$$

Recursively we can find a sequence $n_m \nearrow \infty$ such that $\|b(r-r_{n_m})c\| < \varepsilon_{m+1}$ and so $\|b(r_{n_m}-r_{n_{m-1}})c\| \le 2\varepsilon_m$, $m = 2, 3, \ldots$. Changing notation, we may assume that $\|b(r-r_{n-1})c\| \le \varepsilon_n$ and so $\|b(r_n-r_{n-1})c\| \le 2\varepsilon_n$, $n = 2, 3, \ldots$. Let $c_n = (r_n-r_{n-1})c^2(r_n-r_{n-1})$. Then $\|c_n\| = \|(r_n-r_{n-1})c\|^2 \le 1$.

Case 1. There is N such that $\|c_n\| < \delta_n$ for all $n > N$.

Since $\sum_{n=N+1}^{\infty}\|(r_n-r_{n-1})c\| < \sum_{n=N+1}^{\infty}\delta_n^{1/2} < \infty$, $\sum_{n=1}^{\infty}(r_n-r_{n-1})c$ is a norm convergent sum, i.e., $rc \in C$. Since c is a strictly positive element of C, $r \in M(C)$. Clearly, $r \in M(D)$. Let $e = 0$ and $f = r$.

Case 2. There are infinitely many c_n such that $\|c_n\| \ge \delta_n$.

Since A has FS, all $\mathrm{her}(r_n-r_{n-1})$ $(n \ge 1)$ have FS. For those n with $\|c_n\| \ge \delta_n$ we can find a positive element $h_n \in \mathrm{her}(r_n-r_{n-1})$ with finite spectrum such that $\|h_n - c_n\| < \varepsilon_n$. Then $\|h_n\| > \|c_n\| - \varepsilon_n > \delta_n - \delta_n/6n^2 > \delta_n/2$. By operator calculus, there are mutually orthogonal projections e_{ni} in $\mathrm{her}(r_n-r_{n-1})$ and numbers $t_{ni} \ge 0$ $(1 \le i \le m_n)$ such that $h_n = \sum_{i=1}^{m_n} t_{ni}e_{ni}$.

Since $\|h_n\| > \delta_n/2$, there must be some $t_{ni} \ge \delta_n/2$. Let

$$e_n = \sum_{\{i\,:\,t_{ni}\ge\delta_n/2\}} e_{ni};$$

then $0 \ne e_n \le r_n - r_{n-1}$. Since

$$\left\|b\sum_{\{i\,:\,t_{ni}\ge\delta_n/2\}} t_{ni}e_{ni}\right\| = \left\|b\left[\sum_{\{i\,:\,t_{ni}\ge\delta_n/2\}} t_{ni}^2e_{ni}\right]b\right\|^{1/2}$$
$$\ge (\delta_n/2)\|be_nb\|^{1/2} = (\delta_n/2)\|be_n\|,$$

it follows that

$$\begin{aligned}\|be_n\| &\le (2/\delta_n)\left\|b\sum_{\{i\,:\,t_{ni}\ge\delta_n/2\}} t_{ni}e_{ni}\right\| \le (2/\delta_n)\left\|b\sum_{i=1}^{m_n} t_{ni}e_{ni}\right\| \\ &= (2/\delta_n)\|bh_n\| \le (2/\delta_n)[\|b(h_n-c_n)\| + \|bc_n\|] \\ &\le (2/\delta_n)[\|b\|\,\|h_n-c_n\| + \|b(r_n-r_{n-1})c\|\,\|c\|] \\ &\le (2/\delta_n)[\varepsilon_n + 2\varepsilon_n] = 6\varepsilon_n/\delta_n \le 1/n^2\end{aligned}$$

for those n with $\|c_n\| \ge \delta_n$. Let $d_n = (r_n - r_{n-1} - e_n)c_n(r_n - r_{n-1} - e_n)$. Then

$$\begin{aligned}\|d_n\| &\le \|(r_n - r_{n-1} - e_n)(c_n - h_n)(r_n - r_{n-1} - e_n)\| \\ &\quad + \|(r_n - r_{n-1} - e_n)h_n(r_n - r_{n-1} - e_n)\| \\ &\le \|c_n - h_n\| + \left\|\sum_{\{i\,:\,t_{ni}<\delta_n/2\}} t_{ni}e_{ni}\right\| \\ &< \varepsilon_n + \delta_n/2 \le \delta_n/6n^2 + \delta_n/2 < \delta_n.\end{aligned}$$

Hence $\|(r_n-r_{n-1}-e_n)c\| = \|(r_n-r_{n-1}-e_n)c_n(r_n-r_{n-1}-e_n)\|^{1/2} = \|d_n\|^{1/2} < \delta_n^{1/2}$ and so $\sum_{\{n\,:\,\|c_n\|\ge\delta_n\}}(r_n - r_{n-1} - e_n)c$ is a norm convergent sum. On the other hand, $\sum_{\{n\,:\,\|c_n\|<\delta_n\}}(r_n - r_{n-1})c$ is a norm convergent sum by the same argument as in Case 1.

Define

$$f = \sum_{\{n\,:\,\|c_n\|<\delta_n\}} (r_n - r_{n-1}) + \sum_{\{n\,:\,\|c_n\|\ge\delta_n\}} (r_n - r_{n-1} - e_n).$$

As in Case 1, we can show that $f \in M(C) \cap M(D)$. Let $e = \sum_{\{n\,:\,\|c_n\|\ge\delta_n\}} e_n$. Then $r = e + f$ and $e \in M(B) \cap M(D)$, since

$$\left\|b\sum_{\{n\,:\,\|c_n\|\ge\delta_n\}} e_n\right\| \le \sum_{\{n\,:\,\|c_n\|\ge\delta_n\}} \|be_n\| < \sum_{n=1}^{\infty} 1/n^2 < \infty,$$

and so $be \in B$. □

4.7. *Proof of* (b) ⇒ (d) ⇒ (e). ‘(d) ⇒ (e)’ is trivial.

(b) ⇒ (d). Since A is σ-unital with FS, B and C have FS. By hypothesis together with Proposition 1.2 and [11, 3.34], eBe and fCf have increasing sequential approximate identities $\{e_n\}$ and $\{f_n\}$, respectively, consisting of projections. Let $r_n = e_n + f_n$ for each n. Then $\{r_n\}$ is an approximate identity of D consisting of projections, since $r = e + f$ and $\|d(r - r_n)\| \le \|d(e - e_n)\| + \|d(f - f_n)\| \to 0$ for all d in D. Since $pq = qp = r$, $p + q - r = 1$. Let $p' = 1 - p$ and $q' = 1 - q$. Then p' and q' are closed projections in A^{**}. Let $s = p' + e$. Then $s \in M(C)$, since $s = 1 - p + e = q - r + e = q - f$ and $f \in M(C)$ by hypothesis. Similarly $1 - s = p - e \in M(B)$ and $1 - s = q' + f$. There exist, therefore, increasing approximate identities $\{p'_n\}$ and $\{q'_n\}$ of $(1 - s)B(1 - s)$ and sCs, respectively, consisting of projections. Since $f_n \le 1 - s$ for all n, Lemma 4.1 applies. Then there exists a sequence $m_n \nearrow \infty$ and a unitary

$u \in (1-s)M(B)(1-s)$ such that $f_n \le up'_{m_n}u^*$ for all n. Similarly, $e_n \le s$ $(\forall n)$ implies the existence of $l_n \nearrow \infty$ and a unitary $v \in sM(C)s$ such that $e_n \le vq'_{l_n}v^*$ for all n. Let $p_n = up'_{m_n}u^* + e_n$ and $q_n = vq'_{l_n}v^* + f_n$ for all n. It is clear by construction that $\{p_n\}$ and $\{q_n\}$ are increasing. We claim that $\{p_n\}$ and $\{q_n\}$ are what we want.

First, $p_nq_n = (up'_{m_n}u^* + e_n)(vq'_{l_n}v^* + f_n) = e_n + f_n = r_n$ for all n, since $f_n \le up'_{m_n}u^* \le 1-s$, $e_n \le vq'_{l_n}v^* \le s$, and $e_nf_n = 0$. Similarly $q_np_n = r_n$ for all n. Secondly, for any $b \in B$, $\|b(p-p_n)\| \le \|bu[(1-s)-p'_{m_n}]u^*\| + \|b(e-e_n)\| \to 0$, since $e_n \nearrow e$ and $p'_{m_n} \nearrow 1-s$ in the strict topology of $M(B)$. Hence $\{p_n\}$ is an increasing approximate identity of B consisting of projections. Similarly, $\{q_n\}$ is an increasing approximate identity of C consisting of projections. □

4.8. *Proof of* (e) ⇒ (b).

$$[p-(p_n-p_{n-1})](r_n-r_{n-1})[q-(q_n-q_{n-1})] = p_{n-1}(r_n-r_{n-1})q_{n-1}$$

for each $n \ge 2$ by condition (1). Since $\sum_{n=1}^{\infty}\delta_n < \infty$, we may assume that $\delta_n < 1/4$ for all $n \ge 2$ (delete some terms and rename p_n, q_n, and r_n if necessary).

Let

$$x_n = (r_n-r_{n-1})(q_n-q_{n-1})(r_n-r_{n-1})$$

and

$$y_n = (r_n-r_{n-1})[q-(q_n-q_{n-1})](r_n-r_{n-1}).$$

Then $y_n = (r_n-r_{n-1}) - x_n$ for $n \ge 2$.

Case 1. There is an N such that $\|y_n\| \le \delta_n^2$ if $n \ge N$.

For each $n \ge N$, x_n is invertible in $(r_n-r_{n-1})D(r_n-r_{n-1})$. As in the proof of [20, 2.1], we can choose v_n in her$(r_n-r_{n-1}, q_n-q_{n-1}) \subset C$ such that $v_n^*v_n = r_n-r_{n-1}$, $v_nv_n^* \le q_n-q_{n-1}$, and $\|(r_n-r_{n-1})-v_n\| \le \delta_n+\delta_n^2 < 2\delta_n$. It follows that $\|v_n^*-v_n\| \le \|v_n^*-(r_n-r_{n-1})\| + \|(r_n-r_{n-1})-v_n)\| < 4\delta_n$.

Define $v = \sum_{n=N}^{\infty} v_n + r_{N-1} \in C^{**}$. We show that $v \in M(C)$. In fact, for any $N_2 > N_1 > N$ and any $c \in C$,

$$\left\|c\sum_{n=N_1}^{N_2} v_n\right\|^2 = \left\|c\left[\sum_{n=N_1}^{N_2} v_nv_n^*\right]c^*\right\| \le \left\|c\left[\sum_{n=N_1}^{N_2}(q_n-q_{n-1})\right]c^*\right\|$$
$$\le \|c(q_{N_2}-q_{N_1-1})c^*\| \to 0$$

as $N_1 \to \infty$. On the other hand, since $\sum_{n=1}^{\infty}\delta_n < \infty$, then

$$\left\|\left[\sum_{n=N_1}^{N_2} v_n\right]c\right\| \le \left\|\left[\sum_{n=N_1}^{N_2}(v_n-v_n^*)\right]c\right\| + \left\|\left[\sum_{n=N_1}^{N_2} v_n^*\right]c\right\|$$
$$\le 4\|c\|\left[\sum_{n=N_1}^{N_2}\delta_n\right] + \left\|c^*\left[\sum_{n=N_1}^{N_2} v_n\right]\right\| \to 0$$

as $N_1 \to \infty$. Hence $v \in M(C)$ and so $r = v^*v \in M(C)$. It follows that $r \in M(C)$. It is trivial that $r \in M(D)$. Let $e = 0$ and $f = r$.

Case 2. There are infinitely many n such that $\|y_n\| > \delta_n^2$.

Since A has FS, we can find a positive element $h_n \in \mathrm{her}(r_n - r_{n-1})$ for those n with $\|y_n\| > \delta_n^2$ such that $\|y_n - h_n\| = \gamma_n < \min\{\delta_n^2, (\delta_n^3 - \mu_n)/2\}$, and $\sigma(h_n) = \{t_{ni} : 1 \le i \le m_n\} \subset R^+$. Then $h_n = \sum_{i=1}^{m_n} t_{ni}e_{ni}$ for some mutually orthogonal projections e_{ni} in $\mathrm{her}(r_n - r_{n-1})$.

Let $e_n = \sum_{\{i : t_{ni} > \delta_n^2 - \gamma_n\}} e_{ni}$. Then $0 \ne e_n \le r_n - r_{n-1}$, since $\|h_n\| \ge \|y_n\| - \gamma_n > \delta_n^2 - \gamma_n$. Since

$$\left\|[p - (p_n - p_{n-1})] \sum_{\{i : t_{ni} > \delta_n^2 - \gamma_n\}} t_{ni}e_{ni}\right\| \ge (\delta_n^2 - \gamma_n)\|[p - (p_n - p_{n-1})]e_n\|,$$

we have

$$\begin{aligned}
&\|[p - (p_n - p_{n-1})]e_n\| \\
&\quad \le (\delta_n^2 - \gamma_n)^{-1}\left\|[p - (p_n - p_{n-1})] \sum_{\{i : t_{ni} > \delta_n^2 - \gamma_n\}} t_{ni}e_{ni}\right\| \\
&\quad \le (\delta_n^2 - \gamma_n)^{-1}\|[p - (p_n - p_{n-1})]h_n\| \\
&\quad \le (\delta_n^2 - \gamma_n)^{-1}\{\|h_n - y_n\| + \|[p - (p_n - p_{n-1})]y_n\|\} \\
&\quad = (\delta_n^2 - \gamma_n)^{-1}[\gamma_n + \|p_{n-1}(r_n - r_{n-1})q_{n-1}(r_n - r_{n-1})\|] \\
&\quad \le (\delta_n^2 - \gamma_n)^{-1}(\gamma_n + \mu_n) \\
&\quad < (\delta_n^2 - \gamma_n)^{-1}[\gamma_n + (\delta_n^3 - 2\gamma_n)] = (\delta_n^3 - \gamma_n)/(\delta_n^2 - \gamma_n) < \delta_n
\end{aligned}$$

where $\gamma_n < (\delta_n^3 - \mu_n)/2$ and $\delta_n < 1$. Thus we get

$$\|e_n - e_n(p_n - p_{n-1})e_n\| = \|[p - (p_n - p_{n-1})]e_n\|^2 < \delta_n^2.$$

By the same argument as in Case 1, there are $u_n \in \mathrm{her}(e_n, p_n - p_{n-1}) \subset B$ for those n with $\|y\| > \delta_n^2$ such that $u_n^*u_n = e_n$, $u_nu_n^* \le p_n - p_{n-1}$, and $\|u_n - u_n^*\| < 4\delta_n$. Let $f_n = r_n - r_{n-1} - e_n$. Then $f_n \le q_n$ and

$$\begin{aligned}
\|f_ny_nf_n\| &\le \|f_n(y_n - h_n)f_n\| + \|f_nh_nf_n\| \\
&\le \gamma_n + (\delta_n^2 - \gamma_n) = \delta_n^2.
\end{aligned}$$

It follows that $\|f_n - f_n(q_n - q_{n-1})f_n\| = \|f_nq_{n-1}f_n\| = \|f_ny_nf_n\| \le \delta_n^2$.

By the same argument as in Case 1 again, we can find partial isometries w_n in $\mathrm{her}(f_n, q_n - q_{n-1}) \subset C$ such that

$$w_n^*w_n = f_n, \quad w_nw_n^* \le q_n - q_{n-1}, \quad \text{and} \quad \|w_n - w_n^*\| < 4\delta_n$$

for those n with $\|y_n\| > \delta_n^2$. For those n with $\|y_n\| \le \delta_n^2$, using the same argument as in Case 1 once more we can find partial isometries $v_n \in C$ such that

$$v_n^* v_n = r_n - r_{n-1}, \quad v_n v_n^* \le q_n - q_{n-1}, \quad \text{and} \quad \|v_n^* - v_n\| < 4\delta_n.$$

Define $u = \sum_{\{n\,:\,\|y_n\| > \delta_n^2\}} u_n \in B^{**}$ and

$$v = \sum_{\{n\,:\,\|y_n\| \le \delta_n^2\}} v_n + \sum_{\{n\,:\,\|y_n\| > \delta_n^2 \text{ and } e_n \nleq r_n - r_{n-1}\}} w_n \in C^{**}.$$

We leave it to the reader to check from construction that $u \in M(B)$ and $v \in M(C)$. Let $e = u^* u$ and $f = v^* v$. Therefore, $e \in M(B) \cap M(D)$ and $f \in M(C) \cap M(D)$ are as desired. □

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References

1. C. A. Akemann, *The general Stone-Weierstrass problem*, J. Funct. Anal. **4** (1969), 277–294.
2. ——, *Left ideal structure of C^*-algebras*, J. Funct. Anal. **6** (1970), 305–317.
3. C. A. Akemann, G. K. Pedersen, and J. Tomiyama, *Multipliers of C^*-algebras*, J. Funct. Anal. **13** (1973), 277–301.
4. I. D. Berg, *An extension of the Weyl-von Neumann theorem to normal operators*, Trans. Amer. Math. Soc. **160** (1971), 365–371.
5. J. Bunce and J. Deddens, *A family of simple C^*-algebras related to weighted shift operators*, J. Funct. Anal. **19** (1975), 13–24.
6. B. Blackadar, *Notes on the structure of projections in simple C^*-algebras*, Semesterbericht Funktionalanalysis, W82, Tübingen, March 1983.
7. ——, *K-theory for operator algebras*, Springer-Verlag, New York, 1987.
8. B. Blackadar and A. Kumjian, *Skew products of relations and the structure of simple C^*-algebras*, Math. Z. **189** (1985), 55–63.
9. O. Bratteli, *Inductive limits of finite-dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.
10. L. G. Brown, *Extensions of AF algebras: the projection lifting problem*, Operator Algebras and Applications, Proc. Sympos. Pure Math., vol. 38, part I, Amer. Math. Soc., Providence, R.I., 1981, pp. 175–176.
11. ——, *Semicontinuity and multipliers of C^*-algebras*, Canad. J. Math. **40** (1989), 769–887.
12. ——, private communication.
13. L. G. Brown and G. Pedersen, *C^*-algebras of real rank zero*, preprint.
14. R. Busby, *Double centralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc. **132** (1968), 79–99.
15. M.-D. Choi, *Lifting projections from quotient C^*-algebras*, J. Operator Theory **10** (1983), 21–30.
16. J. Cuntz, *A class of C^*-algebras and topological Markov chains* II: *Reducible Markov chains and the* Ext-*functor for C^*-algebras*, Invent. Math. **63** (1981), 25–40.

17. J. Cuntz and N. Higson, *Kuiper's theorem for Hilbert modules*, Operator Algebra and Mathematical Physics (Proc. Summer Conf. June 17–21, 1985), Contemp. Math., vol. 62, Amer. Math. Soc., Providence, R.I., 1987.

18. J. Dixmier, *C^*-algebras*, North-Holland, 1977.

19. E. G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. in Math., no. 46, Amer. Math. Soc., Providence, R.I., 1981.

20. G. A. Elliott, *Derivations of matroid C^*-àlgebras*. II, Ann. of Math. (2) **100** (1974), 407–422.

21. ——, *Automorphisms determined by multipliers on ideals of a C^*-algebra*, J. Funct. Anal. **23** (1976), 1–10.

22. P. Halmos, *Quasitriangular operators*, Acta Sci. Math. (Szeged) **29** (1968), 283–293.

23. G. J. Murphy, *Diagonality in C^*-algebras*, Math. Z. **199** (1988), 279–284.

24. G. Kasparov, *Hilbert C^*-modules: Theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), 133–150.

25. J. Mingo, *K-theory and multipliers of stable C^*-algebras*, Trans. Amer. Math. Soc. **299** (1987), 397–412.

26. G. K. Pedersen, *The linear span of projections in simple C^*-algebras*, J. Operator Theory **4** (1980), 289–296.

27. ——, *SAW^*-algebras and Corona C^*-algebras, contributions to non-commutative topology*, J. Operator Theory **15** (1986), 15–32.

28. D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), 97–113.

29. H. Weyl, *Ueber beschraenkte quadratischen formen deren differentz vollstetigist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

30. S. Zhang, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras*, Parts I, IV, preprints .

31. ——, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras*, Part II, *K*-Theory (to appear); Part III, Canad. J. Math. **62** (1990), 159–190.

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